Remarks on an example by R. Schoen concerning the scalar curvature

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Abstract:

Classification: 53C21, 53A30

Key words:

1. Introduction

Let M be a compact Riemannian manifold of dimension $n \geq 3$. If g is a Riemannian metric on M, we denote by s_g the scalar curvature and by μ_g the canonical measure of the Riemannian manifold (M, g). Moreover, let $V(g) = \int_M \mu_g$ be the volume, and $S(g) = \int_M s_g \mu_g$ the total scalar curvature of (M, g). We denote by $[g]$ the conformal class of g:

$$[g] = \{fg/f \text{ smooth positive function on } M\},$$

and by $C(M)$ the set of all conformal classes of Riemannian metrics on M. Let $Riem(M)$ be the set of all Riemannian metrics on M. The Einstein-Hilbert functional $\sigma : Riem(M) \to \mathbb{R}$ is given by

$$\sigma(g) = \frac{S(g)}{(V(g))^{\frac{n-2}{n}}}.$$

The volume power in this definition is chosen so that σ is homogeneous. Generally the functional σ is neither bounded from above nor below, but its restriction to every conformal class $C \in C(M)$ is bounded from below; the constant

$$\mu_C = \inf_{g \in C} \sigma(g)$$

is called the Yamabe constant of the class C. Moreover, the infimum is always achieved: there exists a metric $g \in C$ such that $\mu_C = \sigma(g)$; such metrics are called Yamabe metrics. Every Yamabe metric g is a solution of the *Yamabe problem*, that is s_g is constant.

The conformal group of (M, g) is the set of diffeomorphisms ϕ of M such that

$$\forall g' \in [g],\ \phi^* g' \in [g].$$

Let G be a compact subgroup of the conformal group of (M, g), and let

$$\mu_G = \inf_{g' \in [g],\ g'\ G-\text{invariant}} \sigma(g').$$

Then there exists a conformal G-invariant metric g' to g such that

$$\sigma(g') = \mu_G,$$

and so, $s_{g'}$ is constant ([Hebey x]).

2. Position of the problem

Let n be an integer ≥ 3. Let (N, g_N) be a compact connected $n-1$ dimensional Riemannian manifold. We denote its scalar curvature by s_N and its volume by V_N. We suppose that the scalar curvature of this manifold is constant positive, equal to the one of the canonical sphere S^{n-1}, so we have

$$s_N = (n-1)(n-2).$$

On the other hand we consider the circle with length $l > 0$, that is the Riemannian manifold $(S^1(l) = \mathbb{R}/l\mathbb{Z}, dt^2)$, where t denotes the parameter on $S^1(l)$. We take the product manifold $M_l = S^1(l) \times N$ and we consider on M_l the metric g_l given by

$$g_l = dt^2 + g_N.$$

Then we have $V(g_l) = lV_N$, $s_{g_l} = s_N$, and $\sigma(g_l) = s_N (V_N)^{\frac{2}{n}} l^{\frac{2}{n}}$.

Let G_l be the set of isometries of M_l leaving N invariant. Clearly G_l is a compact subgroup of the conformal group of (M_l, g_l) and the set H_l of G_l-invariant metrics that are conformal to g_l is the set of metrics that are conformal to g_l with a ratio depending only on the variable $t \in S^1(l)$:

$$H_l = \{(f \times id_N) g_l / f \in C^\infty(S^1(l), \mathbb{R}_+^*)\}.$$

It follows from the result of Hebey stated in the introduction, that there exists $g' \in H_l$ such that

$$\sigma(g') = \mu_{G_l},$$

and $s_{g'}$ is constant. It follows from the fact that σ is homogeneous that for each $k > 0$ we have

$$\mu_{G_l} = \inf_{g' \in H_l, s_{g'} \equiv k} \sigma(g').$$

We will consider a module of the form $h^{-2} \times id_N$ with $h \in C^\infty(S^1(l), \mathbb{R}_+^*)$ and denote it simply by h^{-2}. Then we have $V(h^{-2}g_l) = V_N \int_0^l h(t)^{-n}\, dt$, so if $s_{h^{-2}g_l}$ is constant we get

$$\sigma(h^{-2}g_l) = s_{h^{-2}g_l}(V_N)^{\frac{2}{n}} \left(\int_0^l h(t)^{-n}\, dt \right)^{\frac{2}{n}}.$$

We will determine all the functions h such that $s_{h^{-2}g_l} \equiv n(n-1)$, and study the behavior of the function

$$\mu : l \mapsto \mu_{G_l}.$$

3. Reduction to an ordinary differential equation and resolution

We consider the metrics of the form $h^{-2}g_l$, where h denotes a positive function on $S^1(l)$. That function h may be considered as a function on $\mathbb{R}$ having period l.

We have $h^{-2}g_l = u^{\frac{4}{n-2}} g_2$, where $u = h^{\frac{2-n}{2}}$ and $g_2 = dt^2 + g_N$. Clearly $s_{g_2} = s_N$. Using (1.161 a)) p.59 in [Besse], we see that the scalar curvatures of $h^{-2}g_l$ and g_2 are related by

$$s_{h^{-2}g_l} = u^{-\frac{4}{n-2}} s_{g_2} - \frac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}} u''.$$

We have $u' = \frac{2-n}{2} h^{-\frac{n}{2}} h'$ and $u'' = \frac{2-n}{2} h^{-\frac{n+2}{2}} \left(hh'' - \frac{n}{2} h'^2 \right)$, hence

$$s_{h^{-2}g_l} = h^2 s_N + (n-1)(2hh'' - nh'^2).$$

We have $s_N = (n-1)(n-2)$, and we impose the condition $s_{h^{-2}g_l} \equiv n(n-1)$. So we get the following differential equation for the module h:

$$(n-2)h^2 + 2hh'' - nh'^2 = n. \tag{1}$$

Of course a first solution is given by the case where h is constant equal to $\sqrt{\frac{n}{n-2}}$; in this case we have

$$\sigma(h^{-2}g_l) = \sigma(g_l) = (n-1)(n-2)(V_N)^{\frac{2}{n}} l^{\frac{2}{n}}.$$

From now on we will look for non-constant solutions h. After multiplication by h' the differential equation (1) becomes $(n-2)h^2h'+h^{n+1}(h^{-n}h'^2)' = nh'$, that is $(n-2)h^{1-n}h'+(h^{-n}h'^2)' = nh^{-n-1}h'$, and finally

$$(-h^{2-n} + h^{-n}h'^2 + h^{-n})' = 0.$$

So there exists a constant K such that $-h^{2-n} + h^{-n}h'^2 + h^{-n} + K = 0$, that is

$$h'^2 = -Kh^n + h^2 - 1. \tag{2}$$

For $K = 0$ the equation becomes $h'^2 = h^2 - 1$ whose solutions $h(t) = \cosh(t + c)$ are not periodic. For $K \neq 0$ we put

$$F_K(s) = -Ks^n + s^2 - 1.$$

Let $K < 0$. Then the derivative ${F_K}'(s) = -nKs^{n-1} + 2s$ is positive for $s > 0$, so equation (2) has no periodic solution.

Now let $K > 0$. Then the derivative ${F_K}'$ vanishes at the point $s_1 = (\frac{2}{nK})^{\frac{1}{n-2}}$. The function F_K increases from -1 to $F_K(s_1)$ on the interval $[0, s_1]$ and decreases from $F_K(s_1)$ to $-\infty$ on the interval $[s_1, +\infty[$. We have

$$F_K(s_1) > 0 \Leftrightarrow K < \frac{2}{n}\left(\frac{n-2}{n}\right)^{\frac{n-2}{2}}.$$

So if $K \geq \frac{2}{n}\left(\frac{n-2}{n}\right)^{\frac{n-2}{2}}$, we have no solution of equation (2). Let $K < \frac{2}{n}\left(\frac{n-2}{n}\right)^{\frac{n-2}{2}}$. Then the function F_K vanishes at two points $0 < s_0 < s_m$, and is positive on the interval $]s_0, s_m[$. Equation (2) is now equivalent to

$$\frac{dh}{\sqrt{F_K(h)}} = \pm dt,$$

that is $\eta_K(h) = \pm t + c$, where

$$\eta_K(h) = \int_{s_0}^{h} \frac{ds}{\sqrt{F_K(s)}}.$$

The function η_K is well defined on the interval $[s_0, s_m]$ since s_0 and s_m are simple roots of the polynomial F_K. We have $\eta_K'(h) = \frac{1}{\sqrt{F_K(h)}}$, so the function η_K increases from 0 to $\eta_K(s_m)$ on the interval $[s_0, s_m]$ and its derivatives at s_0 and s_m are $+\infty$. Hence η_K admits an inverse function η_K^{-1} which increases from s_0 to s_m on the interval $[0, \eta_K(s_m)]$ and whose derivatives at 0 and $\eta_K(s_m)$ vanish. Equation (2) becomes

$$h(t) = \eta_K^{-1}(\pm t + c).$$

It follows that the positive periodic solutions of equation (2) are the functions $h_{K,c} : t \mapsto h_K(t + c)$ where h_K is the even, $2\eta_K(s_m)$-periodic function given by $h_K(t) = \eta_K^{-1}(t)$ for $t \in [0, \eta_K(s_m)]$. We are looking for solutions having period l, that is for functions whose smallest positive period is of the form l/k where k is a positive integer. So the non-constant solutions of our problem are the functions $h_{K,c}$ for which there exists a positive integer k such that

$$\eta_K(s_m) = \frac{l}{2k}. \tag{3}$$

For such solution $h_{K,c}$ we have

$$\sigma({h_{K,c}}^{-2} g_l) = n(n-1)(V_N)^{\frac{2}{n}} \left(\int_0^{2k\eta_K(s_m)} h_K(t)^{-n}\, dt \right)^{\frac{2}{n}}$$

$$= n(n-1)(V_N)^{\frac{2}{n}} (2k)^{\frac{2}{n}} \left(\int_0^{\eta_K(s_m)} h_K(t)^{-n}\, dt \right)^{\frac{2}{n}}$$

$$= n(n-1)(V_N)^{\frac{2}{n}} \left(\frac{l}{\eta_K(s_m)}\right)^{\frac{2}{n}} \left(\int_0^{\eta_K(s_m)} \frac{h_K(t)^{-n} h_K'(t)}{\sqrt{F_K(h_K(t))}}\, dt \right)^{\frac{2}{n}}$$

$$= n(n-1)(V_N)^{\frac{2}{n}} \left(\frac{l}{\eta_K(s_m)}\right)^{\frac{2}{n}} \left(\int_{s_0}^{s_m} \frac{t^{-n}}{\sqrt{F_K(t)}}\, dt \right)^{\frac{2}{n}}.$$

Let $x = K^{\frac{1}{n-2}} s$, $x_0 = K^{\frac{1}{n-2}} s_0$ and $x_m = K^{\frac{1}{n-2}} s_m$. Then $F_K(s) = K^{-\frac{2}{n-2}}(-x^n + x^2 - K^{\frac{2}{n-2}})$, so we get

$$\eta_K(s_m) = \int_{x_0}^{x_m} \frac{dx}{\sqrt{-x^n + x^2 - K^{\frac{2}{n-2}}}}$$

and

$$\int_{s_0}^{s_m} \frac{t^{-n}}{\sqrt{F_K(t)}}\, dt = K^{\frac{n}{n-2}} \int_{x_0}^{x_m} \frac{x^{-n} dx}{\sqrt{-x^n + x^2 - K^{\frac{2}{n-2}}}}.$$

Clearly x_0 and x_m are the roots of the polynomial $-x^n + x^2 - K^{\frac{2}{n-2}}$, so

$$K = ({x_0}^2 - {x_0}^n)^{\frac{n-2}{2}} = ({x_m}^2 - {x_m}^n)^{\frac{n-2}{2}}.$$

The function $x_0 \mapsto K$ is an increasing bijection from $]0, r[$ to $]0, \frac{2}{n}\left(\frac{n-2}{n}\right)^{\frac{n-2}{2}}[$, where

$$r = \left(\frac{2}{n}\right)^{\frac{1}{n-2}}.$$

So, the condition $0 < K < \frac{2}{n}\left(\frac{n-2}{n}\right)^{\frac{n-2}{2}}$ is equivalent to $0 < x_0 < r$. For $x_0 \in]0, r[$ we set

$$P_{x_0}(x) = -x^n + x^2 + {x_0}^n - {x_0}^2$$

and

$$\varphi(x_0)=\int_{x_0}^{x_m}\frac{dx}{\sqrt{P_{x_0}(x)}},\quad \psi(x_0)=({x_0}^2-{x_0}^n)^{\frac{n}{2}}\int_{x_0}^{x_m}\frac{x^{-n}dx}{\sqrt{P_{x_0}(x)}}.$$

Condition (3) becomes

$$\varphi(x_0)=\frac{l}{2k}$$

and we get

$$\sigma(h_{K,c}{}^{-2}g_l)=n(n-1)(V_N)^{\frac{2}{n}}\left(\frac{\psi(x_0)}{\varphi(x_0)}\right)^{\frac{2}{n}}l^{\frac{2}{n}}.$$

4. Variations of the function φ

We have the factorization

$$P_{x_0}(x)=(x-x_0)(x_m-x)Q_{x_0}(x),$$

with

$$Q_{x_0}(x)=x^{n-2}+(x_m+x_0)x^{n-3}+({x_m}^2+x_mx_0+{x_0}^2)x^{n-4}+...$$

$$+({x_m}^{n-3}+{x_m}^{n-4}x_0+...+{x_0}^{n-3})x+({x_m}^{n-2}+{x_m}^{n-3}x_0+...+{x_0}^{n-2}-1)$$

$$=\sum_{k=1}^{n-1}\frac{{x_m}^k-{x_0}^k}{x_m-x_0}x^{n-1-k}-1.$$

Let $x=x_0\cos^2 t+x_m\sin^2 t$. Then $(x-x_0)(x_m-x)=(x_m-x_0)^2\sin^2 t\cos^2 t$, and $dx=(x_m-x_0)\sin 2tdt$, so we get

$$\varphi(x_0)=2\int_0^{\frac{\pi}{2}}\frac{dt}{\sqrt{Q_{x_0}(x_0\cos^2 t+x_m\sin^2 t)}}.$$

We have $\varphi(r)=\frac{\pi}{\sqrt{Q_r(r)}}$, and $Q_r(r)=[1+2+...+(n-1)]r^{n-2}-1=n-2$, so

$$\varphi(r)=\frac{\pi}{\sqrt{n-2}}.$$

Using Fatou's lemma we get $\lim_{x_0\to 0}\varphi(x_0)\geq 2\int_0^{\frac{\pi}{2}}\frac{dt}{\sqrt{Q_0(\sin^2 t)}}$; we have $Q_0(x)=x^{n-2}+x^{n-3}+...+x=x\frac{1-x^{n-2}}{1-x}$, so $\sqrt{Q_0(\sin^2 t)}=\sin t\frac{\sqrt{1-\sin^{2n-4}t}}{\cos t}$, which shows that

$$\lim_{x_0 \to 0} \varphi(x_0) = +\infty.$$

The constant term of the polynomial $Q_{x_0}(x)$ is $Q_{x_0}(0) = \frac{P_{x_0}(0)}{-x_0 x_m} > 0$, so all the coefficients of the polynomial $Q_{x_0}(x)$ are positive. Moreover:

Lemma 1 *The coefficients of the polynomial $Q_{x_0}(x)$ are increasing functions of $x_0 \in [0, r]$.*

Proof We have to show that for every $k \in \{2, 3, ..., n-1\}$, the function

$$F : x_0 \mapsto \frac{{x_m}^k - {x_0}^k}{x_m - x_0}$$

is increasing on $[0, r]$. We denote the derivative of x_m as a function of x_0 by ${x_m}'$. We have

$$F'(x_0) = \frac{(x_m - x_0)(k{x_m}^{k-1}{x_m}' - k{x_0}^{k-1}) - ({x_m}^k - {x_0}^k)({x_m}' - 1)}{(x_m - x_0)^2},$$

so

$$F'(x_0) > 0 \iff [(k-1){x_m}^k - kx_0{x_m}^{k-1} + {x_0}^k]{x_m}' + [(k-1){x_0}^k - k{x_0}^{k-1}x_m + {x_m}^k] > 0$$

$$\iff \big[(k-1) - k\frac{x_0}{x_m} + \big(\frac{x_0}{x_m}\big)^k\big]{x_m}' + [(k-1)\big(\frac{x_0}{x_m}\big)^k - k\big(\frac{x_0}{x_m}\big)^{k-1} + 1] > 0.$$

Let $\alpha = \frac{x_0}{x_m}$. It follows from relation ${x_0}^2 - {x_0}^n = {x_m}^2 - {x_m}^n$ that ${x_m}^{n-2} = \frac{1-\alpha^2}{1-\alpha^n}$; from this we see that the function $\alpha \mapsto x_0$ is an increasing bijection from $[0, 1]$ to $[0, r]$. The preceding condition becomes

$$[(k-1) - k\alpha + \alpha^k]{x_m}' + [(k-1)\alpha^k - k\alpha^{k-1} + 1] > 0.$$

The coefficient of ${x_m}'$ is positive; indeed if we set $u(\alpha) = (k-1) - k\alpha + \alpha^k$, then $u(1) = 0$ and $u'(\alpha) = k(\alpha^{k-1} - 1) < 0$ for $0 < \alpha < 1$. Our condition becomes

$$-{x_m}' < \frac{(k-1)\alpha^k - k\alpha^{k-1} + 1}{(k-1) - k\alpha + \alpha^k}.$$

We denote the second member by $F_k(\alpha)$ and we will show that for every integer $k \geq 1$ we have

$$\forall \alpha \in]0, 1[, F_{k+1}(\alpha) < F_k(\alpha);$$

then it will be sufficient to show that $-{x_m}' < F_{n-1}(\alpha)$. We have

$$F_{k+1}(\alpha) < F_k(\alpha) \iff \frac{k\alpha^{k+1} - (k+1)\alpha^k + 1}{k - (k+1)\alpha + \alpha^{k+1}} < \frac{(k-1)\alpha^k - k\alpha^{k-1} + 1}{(k-1) - k\alpha + \alpha^k}$$

$$\iff -\alpha^{2k+1} + \alpha^{2k} + k^2\alpha^{k+2} + (2-3k^2)\alpha^{k+1} + (3k^2-2)\alpha^k - k^2\alpha^{k-1} - \alpha + 1 > 0$$

$$\iff (1-\alpha) + \alpha^{2k}(1-\alpha) - k^2\alpha^{k-1}(1-\alpha^3) + (3k^2-2)\alpha^k(1-\alpha) > 0$$

$$\iff (1-\alpha)[(\alpha^k-1)^2 - k^2\alpha^{k-1}(\alpha-1)^2] > 0$$

$$\iff (1-\alpha)^3\left[\left(\frac{1-\alpha^k}{1-\alpha}\right)^2 - k^2\alpha^{k-1}\right] > 0$$

$$\iff \frac{1-\alpha^k}{1-\alpha} - k\alpha^{\frac{k-1}{2}} > 0$$

$$\iff 1-\alpha^k - k\alpha^{\frac{k-1}{2}}(1-\alpha) > 0 \tag{4}$$

$$\iff 1-\alpha^k - k\alpha^{\frac{k-1}{2}} + k\alpha^{\frac{k+1}{2}} > 0.$$

Let $f(\alpha) = 1-\alpha^k - k\alpha^{\frac{k-1}{2}} + k\alpha^{\frac{k+1}{2}}$. Then $f'(\alpha) = -k\alpha^{\frac{k-3}{2}}g(\alpha)$ with $g(\alpha) = \alpha^{\frac{k+1}{2}} + \frac{k-1}{2} - \frac{k+1}{2}\alpha$; as $g(1) = 0$ and $g'(\alpha) = \frac{k+1}{2}(\alpha^{\frac{k-1}{2}} - 1) < 0$ we have $g(\alpha) > 0$ for $0 < \alpha < 1$, hence $f'(\alpha) < 0$; as $f(1) = 0$ we have finally that $f(\alpha) > 0$ for $0 < \alpha < 1$.

Let us show now that $-x_m{}' < F_{n-1}(\alpha)$. It follows from relation $x_m{}^n - x_m{}^2 = x_0{}^n - x_0{}^2$ that the derivative $x_m{}'$ satisfies $(nx_m{}^{n-1} - 2x_m)x_m{}' = nx_0{}^{n-1} - 2x_0$, so

$$x_m{}' = \frac{2x_0 - nx_0{}^{n-1}}{2x_m - nx_m{}^{n-1}} = \frac{x_0}{x_m}\ \frac{2-nx_0{}^{n-2}}{2-nx_m{}^{n-2}}$$

$$= \alpha\ \frac{2 - n\alpha^{n-2}\frac{1-\alpha^2}{1-\alpha^n}}{2 - n\frac{1-\alpha^2}{1-\alpha^n}}$$

$$= \alpha\ \frac{(n-2)\alpha^n - n\alpha^{n-2} + 2}{-2\alpha^n + n\alpha^2 - (n-2)}.$$

The denominator is negative because $2x_m - nx_m{}^{n-1} = nx_m(r^{n-2} - x_m{}^{n-2}) < 0$. Hence we get

$$-x_m{}' < F_{n-1}(\alpha) \iff \alpha\ \frac{(n-2)\alpha^n - n\alpha^{n-2} + 2}{2\alpha^n - n\alpha^2 + (n-2)} < \frac{(n-2)\alpha^{n-1} - (n-1)\alpha^{n-2} + 1}{\alpha^{n-1} - (n-1)\alpha + (n-2)}$$

$$\iff \alpha[(n-2)\alpha^n - n\alpha^{n-2} + 2][\alpha^{n-1} - (n-1)\alpha + (n-2)]$$
$$-[(n-2)\alpha^{n-1} - (n-1)\alpha^{n-2} + 1][2\alpha^n - n\alpha^2 + (n-2)] < 0$$

$$\iff -(n-2)(\alpha-1)^2[1-\alpha^{2n-2}-(n-1)\alpha^{n-2}(1-\alpha^2)] < 0.$$

Comparing with (4) we see that the expression between square brackets is positive, which ends the proof of lemma 1.

Proposition 1 *The function φ is decreasing on the interval* $]0,r[$.

Proof We have

$$\varphi(x_0) = 2\int_0^{\frac{\pi}{4}} \left(\frac{1}{\sqrt{Q_{x_0}(x_0\cos^2 t + x_m \sin^2 t)}} + \frac{1}{\sqrt{Q_{x_0}(x_0\sin^2 t + x_m \cos^2 t)}} \right) dt.$$

Hence it is enough to show that for all $t \in]0, \frac{\pi}{4}[$, the function

$$x_0 \mapsto \frac{1}{\sqrt{Q_{x_0}(x_0\cos^2 t + x_m \sin^2 t)}} + \frac{1}{\sqrt{Q_{x_0}(x_0\sin^2 t + x_m \cos^2 t)}}$$

is decreasing on $]0,r[$. For fixed $t \in]0, \frac{\pi}{4}[$ we set

$$u(x_0) = Q_{x_0}(x_0\cos^2 t + x_m \sin^2 t),$$
$$v(x_0) = Q_{x_0}(x_0\sin^2 t + x_m \cos^2 t).$$

It follows from the fact that the coefficients of the polynomial $Q_{x_0}(x)$ are positive, that $0 < u < v$. From lemma 1 we have in particular that ${x_m}' > -1$, so

$$(x_0\cos^2 t + x_m \sin^2 t)' = \cos^2 t + {x_m}' \sin^2 t > \cos^2 t - \sin^2 t > 0,$$

and it follows from lemma 1 that $u' > 0$. We will show that $(u+v)' > 0$. Then we get $(uv)' = u(v' + \frac{v}{u}u') > u(v'+u') > 0$, so uv is increasing, and $\frac{2}{\sqrt{uv}}$ is decreasing; moreover

$$\left(\frac{1}{u} + \frac{1}{v}\right)' = -\frac{1}{v^2}\left(\frac{v^2}{u^2}u' + v'\right) < 0,$$

so $\frac{1}{u} + \frac{1}{v}$ is decreasing; now $(\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{v}})^2 = \frac{1}{u} + \frac{1}{v} + \frac{2}{\sqrt{uv}}$ is decreasing, so $\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{v}}$ is decreasing.

Let us show that $(u+v)' > 0$. Let $x_0 = \alpha x_m$, $a = \alpha\cos^2 t + \sin^2 t$, $b = \cos^2 t + \alpha \sin^2 t$. Then

$$u + v = \frac{1}{\sin^2 t \cos^2 t} \frac{(1+\alpha)(\alpha^n - a^n + 1 - b^n)}{(1-\alpha)(1-\alpha^n)} - \frac{1+\cos(2t)}{\cos^2 t},$$

so it is enough to show that the function

$$f_t : \alpha \mapsto \frac{(1+\alpha)(\alpha^n - a^n + 1 - b^n)}{(1-\alpha)(1-\alpha^n)}$$

is increasing. Straightforward calculation gives the following

$$f_t'(\alpha) = \frac{F_\alpha(t)}{(1-\alpha)^2(1-\alpha^n)^2}$$

where

$$F_\alpha(t) = -2\alpha^{2n} - 2n\alpha^{n+1} + 2n\alpha^{n-1} + 2$$
$$+(a^n+b^n)(n\alpha^{n+1} + 2\alpha^n - n\alpha^{n-1} - 2) + n(a^{n-1}a' + b^{n-1}b')(-\alpha^{n+2} + \alpha^n + \alpha^2 - 1).$$

We have to show that $F_\alpha(t) > 0$ for every $\alpha \in]0,1[$ and every $t \in]0, \frac{\pi}{4}[$. For $t = 0$ we have $a = \alpha$, $b = 1$, $a' = 1$, $b' = 0$, and so

$$F_\alpha(0) = 0.$$

Hence it is enough to show that $F_\alpha'(t) > 0$ for $t \in]0, \frac{\pi}{4}[$. We get

$$F_\alpha'(t) = -n(1-\alpha)^2 \sin(2t) G_\alpha(t)$$

where

$$G_\alpha(t) = [(n-1)\alpha^n + n\alpha^{n-1} + 1](a^{n-1} - b^{n-1})$$
$$+(n-1)(1+\alpha)(1-\alpha^n)(a^{n-2}\cos^2 t - b^{n-2}\sin^2 t);$$

but

$$a^{n-1} - b^{n-1} = a^{n-2}(\alpha\cos^2 t + \sin^2 t) - b^{n-2}(\cos^2 t + \alpha \sin^2 t)$$

$$= \alpha(a^{n-2}\cos^2 t - b^{n-2}\sin^2 t) + (a^{n-2}\sin^2 t - b^{n-2}\cos^2 t),$$

so

$$G_\alpha(t) = a^{n-2}(A\cos^2 t + B\sin^2 t) - b^{n-2}(A\sin^2 t + B\cos^2 t),$$

with

$$A = \alpha^n - n\alpha + n - 1,\ B = (n-1)\alpha^n + n\alpha^{n-1} + 1.$$

We have to show that $G_\alpha(t) < 0$. We have

$$G_\alpha'(t) = \sin(2t)\ G_{\alpha,1}(t),$$

with

$$G_{\alpha,1}(t) = (B-A)(a^{n-2} + b^{n-2})$$
$$+(n-2)(1-\alpha)[a^{n-3}(A\cos^2 t + B\sin^2 t) + b^{n-3}(A\sin^2 t + B\cos^2 t)].$$

We have

$$G'_{\alpha,1}(t) = (n-2)(1-\alpha)\sin(2t)\ G_{\alpha,2}(t),$$

with

$$G_{\alpha,2}(t) = 2(B-A)(a^{n-3} - b^{n-3})$$
$$+(n-3)(1-\alpha)[a^{n-4}(A\cos^2 t + B\sin^2 t) - b^{n-4}(A\sin^2 t + B\cos^2 t)].$$

For $p \in \{1, 2, ..., n-1\}$ let

$$G_{\alpha,p}(t) = p(B-A)(a^{n-1-p} + (-1)^{p+1}b^{n-1-p})$$
$$+(n-1-p)(1-\alpha)[a^{n-2-p}(A\cos^2 t + B\sin^2 t) + (-1)^{p+1}b^{n-2-p}(A\sin^2 t + B\cos^2 t)].$$

Then

$$G'_{\alpha,p}(t) = (n-1-p)(1-\alpha)\sin(2t)\ G_{\alpha,p+1}(t).$$

Clearly $G_\alpha(\frac{\pi}{4}) = 0$, and for even p we have $G_{\alpha,p}(\frac{\pi}{4}) = 0$. We have the following

Lemma 2 *For every $\alpha \in]0,1[$ we have*

a) If $n = 3$, then $G_{\alpha,1}(0) > 0$.
If $n = 4$, then $G_{\alpha,1}(0) = 0$.
If $n \geq 5$, and if p is odd, $p \leq n-2$, then $G_{\alpha,p}(0) < 0$.

b) $G_{\alpha,1}(\pi/4) > 0$.
If p is odd, $\frac{n}{2} \leq p \leq n-1$, then $G_{\alpha,p}(\pi/4) < 0$.
If p is odd, $3 \leq p \leq n-1$, then $[G_{\alpha,p}(\pi/4) \geq 0 \Rightarrow G_{\alpha,p-2}(\pi/4) > 0]$.

c) $G_\alpha(0) < 0$.

Proof a) If p is odd, we have

$$G_{\alpha,p}(0) = p(B-A)(a^{n-1-p} + 1) + (n-1-p)(1-\alpha)(a^{n-2-p}A + B)$$

$$= (n-1)[(p-1)\alpha^{2n-1-p} + (p+1)\alpha^{2n-2-p} - (n-1-p)\alpha^{n+1} + (p-1)\alpha^n + n\alpha^{n-1}$$
$$-n\alpha^{n-p} - (p-1)\alpha^{n-1-p} + (n-1-p)\alpha^{n-2-p} - (p+1)\alpha - (p-1)],$$

in particular

$$G_{\alpha,1}(0) = \alpha(n-1)[2\alpha^{2n-4} - (n-2)\alpha^n + (n-2)\alpha^{n-4} - 2].$$

For $n = 3$ we get $G_{\alpha,1}(0) = 2(1-\alpha^2)(1-\alpha)^2 > 0$. For $n = 4$ we get $G_{\alpha,1}(0) = 0$. Let now $n \geq 5$. First we show that $G_{\alpha,1}(0) < 0$. We have

$$G_{\alpha,1}(0) = \alpha(n-1)f(\alpha)$$

with

$$f(\alpha) = 2\alpha^{2n-4} - (n-2)\alpha^n + (n-2)\alpha^{n-4} - 2.$$

We get

$$f'(\alpha) = (n-2)\alpha^{n-5}h(\alpha)$$

with

$$h(\alpha) = 4\alpha^n - n\alpha^4 + n - 4.$$

We have $h(1) = 0$ and $h'(\alpha) = 4n\alpha^3(\alpha^{n-4} - 1) < 0$, so $h(\alpha) > 0$ and f is increasing; as $f(1) = 0$, we have $f(\alpha) < 0$ and $G_{\alpha,1}(0) < 0$.
Let $G_{\alpha,p}(0) = (n-1)h_{n,p}(\alpha)$, then

$$h_{n+1,p+1}(\alpha) - h_{n,p}(\alpha) = p\ u(\alpha) + n\ v(\alpha) + w(\alpha)$$

with

$$u(\alpha) = \alpha^{2n-p} - \alpha^{2n-2-p} + \alpha^{n+2} - \alpha^n = (\alpha^2 - 1)(\alpha^{2n-2-p} + \alpha^n) < 0,$$

$$v(\alpha) = -\alpha^{n+2} + \alpha^{n+1} + \alpha^n - \alpha^{n-1} = \alpha^{n-1}(1-\alpha)(\alpha^2 - 1) < 0,$$

$$w(\alpha) = 3\alpha^{2n-1-p} - \alpha^{2n-2-p} + \alpha^{n+2} - \alpha^{n+1} + 2\alpha^n - \alpha^{n-p} - \alpha^{n-1-p} - \alpha - 1$$

$$= (\alpha - 1)(\alpha^{2n-2-p} + \alpha^{n+1}) + (\alpha^{n-1} - 1)(\alpha^{n-p} + \alpha) + (\alpha^n - 1)(\alpha^{n-1-p} + 1) < 0,$$

which implies that $h_{n+1,p+1}(\alpha) < h_{n,p}(\alpha)$. We know that $h_{n,1}(\alpha) = \frac{G_{\alpha,1}(0)}{n-1} < 0$ for all $n \geq 5$, $h_{4,1}(\alpha) = 0$, and we have $h_{4,2}(\alpha) = 4\alpha(\alpha+1)(\alpha^2 - 1) < 0$, which ends the proof of a).

b) If p is odd, we get

$$G_{\alpha,p}(\pi/4) = (n-1)\frac{(1+\alpha)^{n-1-p}}{2^{n-2-p}}f(\alpha)$$

with

$$f(\alpha) = -(n-2p)\alpha^n + n\alpha^{n-1} - n\alpha + n - 2p.$$

We have

$$f'(\alpha) = n[-(n-2p)\alpha^{n-1} + (n-1)\alpha^{n-2} - 1],$$

$$f''(\alpha) = n(n-1)\alpha^{n-3}[-(n-2p)\alpha + n - 2].$$

For $p = 1$ we have $f''(\alpha) = n(n-1)(n-2)\alpha^{n-3}(1-\alpha) > 0$ and $f'(1) = 0$, so $f' < 0$ on $]0,1[$; as $f(1) = 0$, we get $f > 0$ on $]0,1[$, that is $G_{\alpha,1}(\pi/4) > 0$.

For $p = n/2$ (in case where $n/2$ is odd) we have $f'(\alpha) = n[(n-1)\alpha^{n-2} - 1]$, so f' vanishes at $\alpha_0 = (n-2)^{-\frac{1}{n-2}}$ and f is decreasing on $[0, \alpha_0]$ and increasing on $[\alpha_0, 1]$; as $f(0) = f(1) = 0$ it follows that $f < 0$ on $]0,1[$, so $G_{\alpha,p}(\pi/4) < 0$.

For $p > n/2$ we have $f'' > 0$ on $]0,1[$; as $f'(0) = -n$ and $f'(1) = 2n(p-1)$, there exists $\alpha_0 \in]0,1[$ such that f is decreasing on $[0, \alpha_0]$ and increasing on $[\alpha_0, 1]$; as $f(0) = n - 2p < 0$ and $f(1) = 0$ it follows that $f < 0$ on $]0,1[$, so $G_{\alpha,p}(\pi/4) < 0$.

Now let us suppose that $G_{\alpha,p}(\pi/4) \geq 0$. We have

$$f(\alpha) = 2p(\alpha^n - 1) - n\alpha^n + n\alpha^{n-1} - n\alpha + n,$$

and this expression is a strictly increasing function of p, from which we conclude that $G_{\alpha,p-2}(\pi/4) > 0$.

c) We have $G_\alpha(0) = \alpha^{n-2}A - B = \alpha^{2n-2} - (n-1)\alpha^n + (n-1)\alpha^{n-2} - 1$. Denoting this expression by $f(\alpha)$ we get $f'(\alpha) = (n-1)\alpha^{n-3}g(\alpha)$ with $g(\alpha) = 2\alpha^n - n\alpha^2 + n - 2$. As $g(1) = 0$ and $g'(\alpha) = 2n\alpha(\alpha^{n-2} - 1) < 0$, we have $g > 0$ on $]0,1[$, and f is increasing. As $f(1) = 0$ we have $f < 0$ on $]0,1[$, so $G_\alpha(0) < 0$.

In order to achieve the proof of proposition 1 we have to show that $G_\alpha(t) < 0$ for all $t \in]0, \pi/4[$, all $\alpha \in]0,1[$ and all integer $n \geq 3$. We have

$$G_{\alpha,n-1}(t) = (n-1)(B-A)[1 + (-1)^n],$$

and

$$B - A = (n-2)(\alpha^n - 1) + n\alpha(\alpha^{n-2} - 1) < 0.$$

Let us consider first the case $n = 3$. Then $G_{\alpha,2}(t) = 0$, so $G_{\alpha,1}(t)$ is constant. By lemma 2 this constant is positive, so $G_\alpha(t)$ is increasing. As $G_\alpha(\pi/4) = 0$, we have $G_\alpha < 0$ on $]0, \pi/4[$.

Now let $n = 4$. Then $G_{\alpha,3}(t) = 6(B-A) < 0$, so $G_{\alpha,2}(t)$ is decreasing. As $G_{\alpha,2}(\pi/4) = 0$, we have $G_{\alpha,2} > 0$ on $]0, \pi/4[$, so $G_{\alpha,1}(t)$ is increasing. As $G_{\alpha,1}(0) = 0$, we have $G_{\alpha,1} > 0$ on $]0, \pi/4[$, so $G_\alpha(t)$ is increasing. As $G_\alpha(\pi/4) = 0$, we have $G_\alpha < 0$ on $]0, \pi/4[$.

Now let $n \geq 5$, n odd. Then $G_{\alpha,n-1} = 0$, so $G_{\alpha,n-2}(t)$ is constant. As $G_{\alpha,n-2}(0) < 0$, this constant is negative. By lemma 2 there exists an odd integer p_0 (depending on α) such that $G_{\alpha,p}(\pi/4) \geq 0$ and such that for every odd integer p we have

$$p_0 < p \leq n-2 \Rightarrow G_{\alpha,p}(\pi/4) < 0$$

and

$$1 \leq p < p_0 \Rightarrow G_{\alpha,p}(\pi/4) > 0.$$

For every odd integer p such that $p_0 < p \leq n-2$ we then have:

$$G_{\alpha,p+1} : \searrow_0,\ G_{\alpha,p+1} > 0,\ G_{\alpha,p} : \nearrow^-,\ G_{\alpha,p} < 0,$$

so

$$G_{\alpha,p_0+1} : \searrow_0,\ G_{\alpha,p_0+1} > 0,\ G_{\alpha,p_0} :\ {}_-\nearrow^+,\ G_{\alpha,p_0} :\ [-,+],$$

and for every odd integer p such that $1 \leq p < p_0$ we have:

$$G_{\alpha,p+1} : \searrow\nearrow^0,\ G_{\alpha,p+1} :\ [+,-],\ G_{\alpha,p} :\ {}_-\nearrow\searrow_+,\ G_{\alpha,p} :\ [-,+],$$

so

$$G_{\alpha,1} :\ [-,+],$$

and finally

$$G_\alpha :\ {}^-\searrow\nearrow^0,$$

so $G_\alpha < 0$ on $]0, \pi/4[$.

For $n \geq 5$, n even, we have $G_{\alpha,n-1} = 2(n-1)(B-A) < 0$, and like before we conclude that $G_\alpha < 0$ on $]0, \pi/4[$.

■

5. Study of the function μ

We have shown in the preceding paragraph that the function φ decreases from $+\infty$ to $\frac{\pi}{\sqrt{n-2}}$ on the interval $]0, r]$. The existence condition (3) for a non-constant solution of our problem stated in paragraph 2 is $\varphi(x_0) = \frac{l}{2k}$, ($k$ integer ≥ 1).

For every $l > 0$ our problem has the constant solution $h_0 \equiv \sqrt{\frac{n}{n-2}}$. It satisfies

$$\sigma_0(l) := \sigma(h_0^{-2} g_l) = (n-1)(n-2)(V_N)^{\frac{2}{n}} l^{\frac{2}{n}}.$$

For $l \leq \frac{2\pi}{\sqrt{n-2}}$ this is the only solution, so

$$\mu(l) = (n-1)(n-2)(V_N)^{\frac{2}{n}} l^{\frac{2}{n}}.$$

Let $k \in \mathbb{N}^*$ and $k\frac{2\pi}{\sqrt{n-2}} < l \leq (k+1)\frac{2\pi}{\sqrt{n-2}}$. Then there exist $x_{0,i}(l) \in]0, r[$, $i = 1, 2, ..., k$, such that $\varphi(x_{0,i}(l)) = \frac{l}{2i}$. The corresponding solutions $h_{l,i}$ satisfy

$$\sigma_i(l) := \sigma({h_{l,i}}^{-2} g_l) = n(n-1)(V_N)^{\frac{2}{n}} \left(\frac{\psi(x_{0,i}(l))}{\varphi(x_{0,i}(l))} \right)^{\frac{2}{n}} l^{\frac{2}{n}}$$

$$= n(n-1)(2iV_N)^{\frac{2}{n}} [\psi(x_{0,i}(l))]^{\frac{2}{n}}.$$

Clearly

$$\mu(l) = \inf_{0 \le i \le k} \sigma_i(l).$$

Proposition 2 *For $l > \frac{2\pi}{\sqrt{n-2}}$ we have $\mu(l) < \sigma_0(l)$.*

Proof It is enough to show that there exists a smooth l-periodic function θ on $\mathbb{R}$ and a (small) real number t such that the function

$$h(x) = 1 + t\theta(x)$$

is positive and satisfies $\sigma(h^{-2}g_l) < \sigma_0(l)$. For small t we have $h^{-n} \sim 1 - nt\theta + \frac{n(n+1)}{2}t^2\theta^2$, so we get (all integrals are taken on the interval $[0,l]$):

$$\int h^{-n} \sim l\left[1 - nt\frac{\int\theta}{l} + \frac{n(n+1)}{2}t^2\frac{\int\theta^2}{l}\right],$$

and

$$\left(\int h^{-n}\right)^{-\frac{n-2}{n}} \sim l\left[1 - nt\frac{\int\theta}{l} + \frac{n(n+1)}{2}t^2\frac{\int\theta^2}{l}\right].$$

Moreover

$$s_{h^{-2}g_l} = h^2 s_N + (n-1)(2hh'' - nh'^2)$$

$$= (n-1)(n-2)(1 + 2t\theta + t^2\theta^2) + (n-1)[2(1+t\theta)t\theta'' - nt^2\theta'^2]$$

$$= (n-1)(n-2)\left[1 + t(2\theta + \tfrac{2}{n-2}\theta'') + t^2(\theta^2 + \tfrac{2}{n-2}\theta\theta'' - \tfrac{n}{n-2}\theta'^2)\right],$$

so we get

$$s_{h^{-2}g_l}h^{-n} \sim (n-1)(n-2)\left\{1 + t\left[(2-n)\theta + \tfrac{2}{n-2}\theta''\right] + t^2\left[\tfrac{n^2-3n+2}{2}\theta^2 + 2\tfrac{n-1}{n-2}\theta\theta'' - \tfrac{n}{n-2}\theta'^2\right]\right\}$$

and

$$\int s_{h^{-2}g_l}h^{-n} \sim (n-1)(n-2)l\left\{1 - (n-2)t\frac{\int\theta}{l} + t^2\left[\frac{n^2-3n+2}{2}\frac{\int\theta^2}{l} + 2\frac{n-1}{n-2}\frac{\int\theta\theta''}{l} - \frac{n}{n-2}\frac{\int\theta'^2}{l}\right]\right\}.$$

We have $\sigma(h^{-2}g_l) = (V_N)^{\frac{2}{n}}\left(\int s_{h^{-2}g_l}h^{-n}\right)\left(\int h^{-n}\right)^{-\frac{n-2}{n}}$, so

$$\sigma(h^{-2}g_l) \sim \sigma_0(l)[1 + \alpha(\theta)t^2]$$

with

$$\alpha(\theta) = \tfrac{1}{l^2}\big\{(n-2)\big[(\textstyle\int\theta)^2 - l\int\theta^2\big] + l\int\theta'^2\big\}.$$

Let $\theta(x) = \sin(\frac{2\pi}{l}x)$, then $\alpha(\theta) = \frac{2-n}{2} + \frac{2\pi^2}{l^2}$, so $\alpha(\theta) < 0 \iff l > \frac{2\pi}{\sqrt{n-2}}$. Hence this function θ satisfies our conditions.

■

It follows from proposition 2 that for $\frac{2\pi}{\sqrt{n-2}} < l \leq 2\frac{2\pi}{\sqrt{n-2}}$ we have $\mu(l) = \sigma_1(l)$. If $l \to \frac{2\pi}{\sqrt{n-2}}$, $(l > \frac{2\pi}{\sqrt{n-2}})$, then $x_{0,1}(l) = \varphi^{-1}(\frac{l}{2}) \to \varphi^{-1}(\frac{\pi}{\sqrt{n-2}}) = r$, and

$$\sigma_1(l) \to n(n-1)(V_N)^{\frac{2}{n}} 2^{\frac{2}{n}} (\psi(r))^{\frac{2}{n}}$$

$$= n(n-1)(V_N)^{\frac{2}{n}} 2^{\frac{2}{n}} (r^2 - r^n)(\tfrac{\pi}{r^n\sqrt{n-2}})^{\frac{2}{n}}$$

$$= n(n-1)(V_N)^{\frac{2}{n}} \big(1 - \tfrac{2}{n}\big)(\tfrac{2\pi}{\sqrt{n-2}})^{\frac{2}{n}}$$

$$= (n-1)(n-2)(V_N)^{\frac{2}{n}} (\tfrac{2\pi}{\sqrt{n-2}})^{\frac{2}{n}}$$

$$= \sigma_0(\tfrac{2\pi}{\sqrt{n-2}}),$$

which shows that the function μ is continuous at the point $\frac{2\pi}{\sqrt{n-2}}$.

Proposition 3 *The function μ is differentiable at the point $\frac{2\pi}{\sqrt{n-2}}$.*

Proof For $l < \frac{2\pi}{\sqrt{n-2}}$ we have $\mu'(l) = \frac{2(n-1)(n-2)}{n}(V_n)^{\frac{2}{n}} l^{\frac{2-n}{n}}$, so

$$\mu'_{left}(\tfrac{2\pi}{\sqrt{n-2}}) = \tfrac{(n-1)(n-2)}{n}(2V_n)^{\frac{2}{n}}(\tfrac{\pi}{\sqrt{n-2}})^{\frac{2-n}{n}}.$$

For $l > \frac{2\pi}{\sqrt{n-2}}$ we have $\mu(l) = \sigma_1(l) = n(n-1)(2V_n)^{\frac{2}{n}}(\psi(\varphi^{-1}(\frac{l}{2})))^{\frac{2}{n}}$, so

$$\mu'(l) = (n-1)(2V_n)^{\frac{2}{n}}(\psi(\varphi^{-1}(\tfrac{l}{2})))^{\frac{2-n}{n}} \frac{\psi'(\varphi^{-1}(\frac{l}{2}))}{\varphi'(\varphi^{-1}(\frac{l}{2}))},$$

and

$$\mu'_{right}(\tfrac{2\pi}{\sqrt{n-2}}) = (n-1)(2V_n)^{\frac{2}{n}}(\psi(r))^{\frac{2-n}{n}} \lim_{s\to r}\frac{\psi'(s)}{\varphi'(s)}.$$

From $x_0^2 - x_m^2 = x_0^n - x_m^n$ and $x_0 \neq x_m$ we get $x_0 + x_m = \sum_{i=o}^{n-1} x_0^{n-1-i} x_m^i$, so

$$1 + x_m' = \sum_{i=o}^{n-1}[(n-1-i)x_0^{n-2-i}x_m^i + i x_0^{n-1-i}x_m^{i-1}x_m'],$$

from which we get $1 + x'_m(r) = (n-1)[1 + x'_m(r)]$, so

$$x'_m(r) = -1.$$

Let $x_0 = r - h$, then $x_m = r + h + ah^2 + bh^3 + ch^4 + ...$ Putting this in relation $x_0^2 - x_m^2 = x_0^n - x_m^n$, we get the values of the constants a, b, c:

$$x_m = r + h - \tfrac{n-1}{3r}h^2 + (\tfrac{n-1}{3r})^2 h^3 - \tfrac{(n-1)(19n^2-23n+58)}{540r^3}h^4 + ...$$

Let $x = x_0 \cos^2 t + x_m \sin^2 t$. Then we get after some calculations

$$Q_{x_0}(x) = (n-2)\{1 - \tfrac{n-1}{3r}\cos 2t \cdot h - \tfrac{n-1}{36r^2}[12 - 2(n-1)\cos 2t - 3(n-3)\cos^2 2t]h^2\} + ...,$$

$$\frac{1}{\sqrt{Q_{x_0}(x)}} = \frac{1}{\sqrt{n-2}}\{1 + \tfrac{n-1}{6r}\cos 2t \cdot h + \tfrac{n-1}{72r^2}[3(n+2) - 2(n-1)\cos 2t + 3\cos 4t]h^2\} + ...,$$

$$\varphi(x_0) = \frac{\pi}{\sqrt{n-2}}[1 + \tfrac{(n-1)(n+2)}{24r^2}h^2] + ...,$$

$$\psi(x_0) = \frac{\pi}{\sqrt{n-2}}(\tfrac{n-2}{n})^{\frac{n}{2}}[1 + \tfrac{(n-1)(n+2)}{24r^2}h^2] + ...,$$

so

$$\mu'_{right}(\tfrac{2\pi}{\sqrt{n-2}}) = (n-1)(2V_n)^{\frac{2}{n}}(\tfrac{n-2}{n})^{\frac{2-n}{2}}(\tfrac{\pi}{\sqrt{n-2}})^{\frac{2-n}{n}}\frac{\psi''(r)}{\varphi''(r)}$$

$$= (n-1)(2V_n)^{\frac{2}{n}}(\tfrac{n-2}{n})^{\frac{2-n}{2}}(\tfrac{\pi}{\sqrt{n-2}})^{\frac{2-n}{n}}(\tfrac{n-2}{n})^{\frac{n}{2}}$$

$$= \tfrac{(n-1)(n-2)}{n}(2V_n)^{\frac{2}{n}}(\tfrac{\pi}{\sqrt{n-2}})^{\frac{2-n}{n}}$$

$$= \mu'_{left}(\tfrac{2\pi}{\sqrt{n-2}}).$$

Proposition 4 *For $l > \frac{2\pi}{\sqrt{n-2}}$ we have $\sigma_1(l) < \sigma_0(l)$.*

Proof As $\sigma_1(l) = n(n-1)(2V_N)^{\frac{2}{n}}(\psi(x_0))^{\frac{2}{n}}$ and $\sigma_0(l) = (n-1)(n-2)(2V_N)^{\frac{2}{n}}(\varphi(x_0))^{\frac{2}{n}}$, with $x_0 = \varphi^{-1}(\frac{l}{2})$, we have to show that for all $x_0 \in]0, r[$,

$$\psi(x_0) < (1 - \tfrac{2}{n})^{\frac{n}{2}}\varphi(x_0).$$

Let us recall that $\varphi(x_0) = \int_{x_0}^{x_m} \frac{dx}{\sqrt{P_{x_0}(x)}}$ and $\psi(x_0) = ({x_0}^2 - {x_0}^n)^{\frac{n}{2}} \int_{x_0}^{x_m} \frac{x^{-n}dx}{\sqrt{P_{x_0}(x)}}$. Let $s = \sqrt{\frac{x^2}{x_0^2 - x_0^n} - 1}$, $s_0 = \sqrt{\frac{x_0^{n-2}}{1-x_0^{n-2}}}$, $s_m = \sqrt{\frac{x_m^{n-2}}{1-x_m^{n-2}}}$; then

$$\varphi(x_0) = \int_{s_0}^{s_m} \frac{ds}{(1+s^2)^{\frac{1}{2}}\sqrt{1-(x_0^2-x_0^n)^{\frac{n-2}{2}}\frac{(1+s^2)^{\frac{n}{2}}}{s^2}}}$$

$$= \sum_{p=0}^{+\infty} \frac{(2p)!}{2^{2p}(p!)^2}(x_0^2-x_0^n)^{p\frac{n-2}{2}} \int_{s_0}^{s_m} \frac{(1+s^2)^{\frac{pn-1}{2}}}{s^{2p}} ds,$$

$$\psi(x_0) = \int_{s_0}^{s_m} \frac{ds}{(1+s^2)^{\frac{n+1}{2}}\sqrt{1-(x_0^2-x_0^n)^{\frac{n-2}{2}}\frac{(1+s^2)^{\frac{n}{2}}}{s^2}}}$$

$$= \sum_{p=0}^{+\infty} \frac{(2p)!}{2^{2p}(p!)^2}(x_0^2-x_0^n)^{p\frac{n-2}{2}} \int_{s_0}^{s_m} \frac{(1+s^2)^{\frac{(p-1)n-1}{2}}}{s^{2p}} ds,$$

so it is enough to show that for all integer $p \geq 0$ and all $x_0 \in]0, r[$ we have

$$\int_{s_0}^{s_m} \frac{(1+s^2)^{\frac{(p-1)n-1}{2}}}{s^{2p}} ds < (1-\tfrac{2}{n})^{\frac{n}{2}} \int_{s_0}^{s_m} \frac{(1+s^2)^{\frac{pn-1}{2}}}{s^{2p}} ds.$$

Let

$$F_p(x_0) = (1-\tfrac{2}{n})^{\frac{n}{2}} \int_{s_0}^{s_m} \frac{(1+s^2)^{\frac{pn-1}{2}}}{s^{2p}} ds - \int_{s_0}^{s_m} \frac{(1+s^2)^{\frac{(p-1)n-1}{2}}}{s^{2p}} ds.$$

If $x_0 = r$, then $s_0 = s_m$, so $F(r) = 0$. Hence it is enough to show that the function F is strictly decreasing on $]0, r[$. we get

$$\tfrac{2}{n-2}(x_0^2-x_0^n)^{1+\frac{p(n-2)}{2}} F_p'(x_0) = [(1-\tfrac{2}{n})^{\frac{n}{2}}-(1-x_m^{n-2})^{\frac{n}{2}}]x_m^{\frac{n}{2}}x_m' - [(1-\tfrac{2}{n})^{\frac{n}{2}}-(1-x_0^{n-2})^{\frac{n}{2}}]x_0^{\frac{n}{2}},$$

so we have to show that

$$x_m' < \frac{x_0^{\frac{n}{2}}}{x_m^{\frac{n}{2}}} \frac{(1-\frac{2}{n})^{\frac{n}{2}}-(1-x_0^{n-2})^{\frac{n}{2}}}{(1-\frac{2}{n})^{\frac{n}{2}}-(1-x_m^{n-2})^{\frac{n}{2}}},$$

i.e.

$$\frac{x_0}{x_m} \frac{(1-\frac{2}{n})-(1-x_0^{n-2})}{(1-\frac{2}{n})-(1-x_m^{n-2})} < \left(\frac{x_0}{x_m}\right)^{\frac{n}{2}} \frac{(1-\frac{2}{n})^{\frac{n}{2}}-(1-x_0^{n-2})^{\frac{n}{2}}}{(1-\frac{2}{n})^{\frac{n}{2}}-(1-x_m^{n-2})^{\frac{n}{2}}}. \tag{5}$$

Let $\alpha = \frac{x_0}{x_m}$ and $u = \sqrt{\frac{n}{n-2}\frac{1-\alpha^{n-2}}{1-\alpha^n}}$. Clearly $u > 1$ and $\alpha u < 1$. Moreover we have $\alpha^{1/2}u < 1$. Inequality (5) becomes

$$\frac{1-u^2}{1-(\alpha u)^2} < \alpha^{\frac{n}{2}-1}\frac{1-u^n}{1-(\alpha u)^n},$$

i.e.

$$\left(\sum_{k=0}^{n-1}(\alpha u)^k\right)(1+u)-\alpha^{\frac{n}{2}-1}(1+\alpha u)\left(\sum_{k=0}^{n-1}u^k\right)>0. \tag{6}$$

For even n this inequality reads

$$\left(\sum_{k=0}^{\frac{n-2}{2}}[(\alpha u)^k+(\alpha u)^{n-1-k}]\right)(1+u)-\alpha^{\frac{n}{2}-1}(1+\alpha u)\left(\sum_{k=0}^{\frac{n-2}{2}}(u^k+u^{n-1-k})\right)>0.$$

For each integer k such that $0\le k\le \frac{n-2}{2}$ we have

$$[(\alpha u)^k+(\alpha u)^{n-1-k}](1+u)-\alpha^{\frac{n}{2}-1}(1+\alpha u)(u^k+u^{n-1-k})$$

$$=(\alpha u)^k\big[(1+(\alpha u)^{n-1-2k})(1+u)-\alpha^{\frac{n}{2}-1-k}(1+\alpha u)(1+u^{n-1-2k})\big]$$

$$=(\alpha u)^k\big[(1-\alpha^{\frac{n}{2}-1-k})(1-(\alpha^{1/2}u)^{n-2k})+u(1-\alpha^{\frac{n}{2}-k})(1-(\alpha^{1/2}u)^{n-2-2k})\big],$$

which is positive as $\alpha^{1/2}u<1$. This proves inequality (6) in the case n even. For odd n this inequality reads

$$\left((\alpha u)^{\frac{n-1}{2}}+\sum_{k=0}^{\frac{n-3}{2}}[(\alpha u)^k+(\alpha u)^{n-1-k}]\right)(1+u)$$

$$-\alpha^{\frac{n}{2}-1}(1+\alpha u)\left(u^{\frac{n-1}{2}}+\sum_{k=0}^{\frac{n-3}{2}}(u^k+u^{n-1-k})\right)>0,$$

i.e.

$$(\alpha u)^{\frac{n-1}{2}}(1+u)-\alpha^{\frac{n}{2}-1}(1+\alpha u)u^{\frac{n-1}{2}}$$

$$+\sum_{k=0}^{\frac{n-3}{2}}\left((\alpha u)^k\big[(1-\alpha^{\frac{n}{2}-1-k})(1-(\alpha^{1/2}u)^{n-2k})+u(1-\alpha^{\frac{n}{2}-k})(1-(\alpha^{1/2}u)^{n-2-2k})\big]\right)>0.$$

The first term is negative:

$$(\alpha u)^{\frac{n-1}{2}}(1+u)-\alpha^{\frac{n}{2}-1}(1+\alpha u)u^{\frac{n-1}{2}}=-\alpha^{\frac{n}{2}-1}u^{\frac{n-1}{2}}(1-\alpha^{1/2})(1-\alpha^{1/2}u),$$

but it is compensated by the first term in the sum:

$$-\alpha^{\frac{n}{2}-1}u^{\frac{n-1}{2}}(1-\alpha^{1/2})(1-\alpha^{1/2}u)+(1-\alpha^{\frac{n}{2}-1})(1-\alpha^{\frac{n}{2}}u^n)$$

$$=(1-\alpha^{1/2})(1-\alpha^{1/2}u)\Big[\Big(\sum_{j=0}^{n-3}\alpha^{j/2}\Big)\Big(\sum_{j=0}^{n-1}(\alpha^{1/2}u)^k\Big)-\alpha^{\frac{n}{2}-1}u^{\frac{n-1}{2}}\Big];$$

as $\alpha^{\frac{n}{2}-1}u^{\frac{n-1}{2}}=\alpha^{(\frac{n-3}{2})/2}(\alpha^{1/2}u)^{\frac{n-1}{2}}$, the expression between brackets is positive, which ends the proof.

Remark 1 Let k be an integer ≥ 1 and let $l>k\frac{2\pi}{\sqrt{n-2}}$.
There exists a unique $x_{0,k}(l)\in]0,r[$ such that $\varphi(x_{0,k})(l)=\frac{l}{2k}$, and there exists a unique $x_{0,1}(l/k)\in]0,r[$ such that $\varphi(x_{0,1})(l/k)=\frac{l}{2k}$, so

$$x_{0,k}(l)=x_{0,1}(l/k).$$

It follows that

$$\sigma_k(l)=k^{\frac{2}{n}}\sigma_1(l/k). \tag{7}$$

By proposition 4 we have $k^{\frac{2}{n}}\sigma_1(l/k)<k^{\frac{2}{n}}\sigma_0(l/k)=\sigma_0(l)$, so

$$\sigma_k(l)<\sigma_0(l). \tag{8}$$

If $l\to k\frac{2\pi}{\sqrt{n-2}}$, then $x_{0,k}(l)=\varphi^{-1}(\frac{l}{2k})\to\varphi^{-1}(\frac{\pi}{\sqrt{n-2}})=r$, and

$$\sigma_k(l)\to\sigma_0(k\frac{2\pi}{\sqrt{n-2}}). \tag{9}$$

If $l\to+\infty$, then $x_{0,k}(l)\to 0$, and

$$\sigma_k(l)\to n(n-1)(2kV_N)^{\frac{2}{n}}[\psi(0)]^{\frac{2}{n}}. \tag{9}$$

In particular $\sigma_k(+\infty)=k^{\frac{2}{n}}\sigma_1(+\infty)$.

The following proposition shows that $\psi(0)$ is finite.

Proposition 5 *Let* $\psi_n(x_0)=({x_0}^2-{x_0}^n)^{\frac{n}{2}}\int_{x_0}^{x_m}\frac{x^{-n}dx}{\sqrt{P_{x_0}(x)}}$.
If n is even, $n=2p$, we have

$$\lim_{x_0\to 0}\psi_n(x_0)=\frac{2^{2p-2}[(p-1)!]^2}{(2p-1)!}.$$

If n is odd, $n=2p+1$, we have

$$\lim_{x_0\to 0}\psi_n(x_0)=\frac{(2p)!\pi}{2^{2p+1}(p!)^2}.$$

Proof We write $\psi(x_0)$ in the form

$$\psi(x_0) = (1 - x_0^{n-2})^{\frac{n}{2}} x_0^n \Big(\int_{x_0}^{\sqrt{x_0}} \frac{x^{-n} dx}{\sqrt{P_{x_0}(x)}} + \int_{\sqrt{x_0}}^{r} \frac{x^{-n} dx}{\sqrt{P_{x_0}(x)}} + \int_{r}^{x_m} \frac{x^{-n} dx}{\sqrt{P_{x_0}(x)}} \Big).$$

For $\sqrt{x_0} \le x \le r$ we have $x^{-n} \le {x_0}^{-n/2}$, and as the function $x \mapsto P_{x_0}(x)$ is increasing on $[0, r]$, $P_{x_0}(x) \ge P_{x_0}(\sqrt{x_0}) = x_0 - x_0^{n/2} - x_0^2 + x_0^n$, so

$$x_0^n \int_{\sqrt{x_0}}^{r} \frac{x^{-n} dx}{\sqrt{P_{x_0}(x)}} \le x_0^{\frac{n-1}{2}} \frac{r - \sqrt{x_0}}{\sqrt{1 - x_0^{\frac{n}{2}-1} - x_0 + x_0^{n-1}}},$$

which shows that

$$\lim_{x_0 \to 0} x_0^n \int_{\sqrt{x_0}}^{r} \frac{x^{-n} dx}{\sqrt{P_{x_0}(x)}} = 0.$$

As $\int_r^{x_m} \frac{dx}{\sqrt{P_{x_0}(x)}}$ is bounded when $x_0 \to 0$, we have

$$\lim_{x_0 \to 0} x_0^n \int_{r}^{x_m} \frac{x^{-n} dx}{\sqrt{P_{x_0}(x)}} = 0.$$

We have $P_{x_0}(x) = (x^2 - x_0^2)\big(1 - \frac{x^n - x_0^n}{x^2 - x_0^2}\big)$, and for $x_0 < x < \sqrt{x_0}$,

$$\tfrac{n}{2} x_0^{n-2} < \frac{x^n - x_0^n}{x^2 - x_0^2} < \frac{x_0^{n/2} - x_0^n}{x_0 - x_0^2},$$

so

$$\frac{1}{\sqrt{1 - \frac{n}{2} x_0^{n-2}}} < \frac{1}{\sqrt{1 - \frac{x^n - x_0^n}{x^2 - x_0^2}}} < \frac{1}{\sqrt{1 - \frac{x_0^{\frac{n}{2}-1} - x_0^{n-1}}{1 - x_0}}}.$$

It follows that

$$\lim_{x_0 \to 0} \psi(x_0) = \lim_{x_0 \to 0} x_0^n \int_{x_0}^{\sqrt{x_0}} \frac{x^{-n} dx}{\sqrt{x^2 - x_0^2}}.$$

Let $t = \sqrt{x^2 - x_0^2}$, then

$$\int_{x_0}^{\sqrt{x_0}} \frac{x^{-n} dx}{\sqrt{x^2 - x_0^2}} = \int_0^{\sqrt{x_0 - x_0^2}} \frac{dt}{(t^2 + x_0^2)^{\frac{n+1}{2}}},$$

and standard calculations show the announced results.

Remark 2 The function $n \mapsto \psi_n(0)$ is decreasing, so we have for all integer $n \ge 3$, $\psi_n(0) \le \psi_n(3) = \pi/4$. Moreover $\lim_{n \to \infty} \psi_n(0) = 0$.

Proposition 6 *Let $n = 4$. Then*

$$\varphi(x_0) = \frac{\pi\sqrt{2}}{2} \sum_{p=0}^{+\infty} \frac{(4p)!}{2^{6p}(p!)^2(2p)!}(x_m^2 - x_0^2)^{2p},$$

$$\psi(x_0) = \frac{\pi\sqrt{2}}{8} \sum_{p=0}^{+\infty} \frac{(4p)!}{2^{6p}(p!)^2(2p)!} \frac{3}{(4p-3)(4p-1)} (x_m^2 - x_0^2)^{2p},$$

which shows in particular that the function ψ is decreasing.

Proof In this case we have $x_0^2 + x_m^2 = 1$ and $0 < x_0 < r = \frac{1}{\sqrt{2}} < x_m < 1$, so there exists $\theta \in]0, \frac{\pi}{4}[$ such that $x_0 = \sin\theta$ and $x_m = \cos\theta$. Let

$$x^2 = x_0^2 \cos^2 t + x_m^2 \sin^2 t;$$

then $2xdx = (x_m^2 - x_0^2)\sin 2tdt$, $P_{x_0}(x) = (x^2 - x_0^2)(x_m^2 - x^2) = \frac{1}{4}(x_m^2 - x_0^2)^2 \sin^2 2t$, $x^2 = \frac{1}{2}(1 - \cos 2\theta cos 2t)$, so

$$\begin{aligned}\psi(x_0) &= (x_0^2 - x_0^4)^2 \int_{x_0}^{x_m} \frac{dx}{x^4\sqrt{P_{x_0}(x)}} \\ &= \frac{\sqrt{2}}{4}\sin^4 2\theta \int_0^{\pi/2} \frac{dt}{(1 - \cos 2\theta \cos 2t)^{5/2}} \\ &= \frac{\sqrt{2}}{8}(1 - \cos^2 2\theta)^2 \int_0^{\pi} \frac{dt}{(1 - \cos 2\theta \cos t)^{5/2}}.\end{aligned}$$

Let $s = \cos 2\theta = x_m^2 - x_0^2$; then

$$\begin{aligned}\psi(x_0) &= \frac{\sqrt{2}}{8}(1 - s^2)^2 \int_0^{\pi} \left(\sum_{p=0}^{+\infty}(-1)^p \frac{(2p+4)!}{3 \cdot 2^{2p+2}(p+2)!p!} s^p \cos^p t\right) dt \\ &= \frac{\sqrt{2}}{8}(1 - s^2)^2 \sum_{p=0}^{+\infty}(-1)^p \frac{(2p+4)!}{3 \cdot 2^{2p+2}(p+2)!p!} s^p \int_0^{\pi} \cos^p tdt \\ &= \frac{\sqrt{2}}{8}(1 - s^2)^2 \sum_{p=0}^{+\infty} \frac{(4p+4)!}{3 \cdot 2^{4p+2}(2p+2)!(2p)!} s^{2p} \int_0^{\pi} \cos^{2p} tdt \\ &= \frac{\sqrt{2}}{8}(1 - s^2)^2 \sum_{p=0}^{+\infty} \frac{(4p+4)!}{3 \cdot 2^{4p+2}(2p+2)!(2p)!} s^{2p} \frac{(2p)!}{2^{2p}(p!)^2}\pi \\ &= \frac{\pi\sqrt{2}}{8}(1 - s^2)^2 \sum_{p=0}^{+\infty} \frac{(4p+4)!}{3 \cdot 2^{6p+2}(2p+2)!(p!)^2} s^{2p}\end{aligned}$$

$$= \frac{\pi\sqrt{2}}{8} \sum_{p=0}^{+\infty} \frac{(4p)!}{2^{6p}(p!)^2(2p)!} \; \frac{3}{(4p-3)(4p-1)} s^{2p},$$

and the formula for $\varphi(x_0)$ is obtained in the same way.